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Alternating bar instabilities in unsteady channel flows over erodible beds

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Channel flows over erodible beds are susceptible to instabilities of the coupled fluid and sediment flow equations. The most dangerous mode usually takes the form of a migrating alternating bar instability propagating in the flow direction. Previous theories have assumed that the underlying flow is steady and here the theory is extended to the unsteady situation. Stability characteristics are calculated for large-amplitude oscillations superimposed on a mean flow. In addition it is found that the basic instability is convective and we address the receptivity problem for bars induced by flow oscillations interacting with spatial variations associated with seepage, channel width variations etc. The effect of unsteadiness in the weakly nonlinear situation is also discussed. A mechanism which allows flow oscillations to interact with migrating bars to produce a sinusoidal structure fixed in space which might be relevant to meander formation is discussed.

1. Introduction

Our concern is the role of flow unsteadiness in the formation of alternating bars in channel flows. We will restrict ourselves to time variations on the scale of the transport of sediment and will further confine our attention here to the case when the time variation is periodic though the analysis is valid for more general time dependences. In the periodic case Floquet theory gives a natural definition of instability and we are not confined to restricted sizes of the amplitude and frequency of the unsteady part of the flow.

It has been known for a long time that the flow in a channel with an erodible bed is unstable to alternating bars which propagate in the flow direction and produce distinct fronts separating the higher and lower regions of the bed. The reader is referred to Columbini, Seminara & Tubino (1987, hereafter referred to as CST), for a discussion of the early linear theories of bar formation. The formation of bars has important consequences for the navigation of the channel. Indeed, as pointed out to the author by a referee, bars are believed to be the fundamental morphodynamic building block required to form meanders and braided beds and therefore have important consequences for river ecology and river training.

Various proposals have been made about the relationship between bar formation and meanders in rivers. See, for example Blondeaux & Seminara (1985, hereafter referred to as BS), for a discussion. The main difficulty in the identification of bars as a source for the generation of meanders in rivers is the fact that almost all of the unstable wavenumbers correspond to bars migrating in the flow direction whereas a spatially fixed alternating bar naturally produces high shear stresses over the ‘pools’, leading to a migration of the channel. Thus, without being able to fix bars spatially, there is no obvious mechanism through which they can produce meanders; we will argue that flow unsteadiness is a possible means to fix the bars. The most popular model for the evolution of meanders is that given by Johannesson & Parker (1989). The planimetric evolution equation given in that paper has formed the basis for a series of papers on meanders by for example Parker & Andrews (1986), Furbish (1988) and Howard (1996). Note however that many of the ideas given in Johannesson & Parker (1989) can be found in BS. More recently a simpler meander evolution equation has been proposed by Zolezzi & Seminara (2001). At very small wavenumbers there is a zero-frequency mode which corresponds to stationary bars but it has a wavelength much larger than typical meander wavelengths and BS show how this mode crucially interacts in a resonant manner with a streamwise variation induced by an existing meander. However it appears that there is no general agreement on whether this resonant interaction can lead to the formation of meanders.

Here we will attempt to shed light on the linear and nonlinear stages of bar development in unsteady flows. The unsteady variations which we consider take place on the sediment transport timescale. Seasonal variations in channel depth and flow speed for the gravel bed rivers considered here take place on a somewhat longer timescale, but, since our results will be seen to be independent of the frequency of the variations, our theory is relevant to that case. It should also be noted that our analysis can be carried over to sandy beds where time-periodic variations on the sediment timescale are relevant, so it is worthwhile to study the periodic case. In addition, our analysis is relevant to controlled floods where the flow rate is varied on the sediment transport timescale. In the linear regime we will consider time-periodic channel flows where the oscillatory component is comparable with the mean flow speed and we determine how the critical channel width varies as a function of the amplitude of the unsteady part of the flow. In order to carry out the stability analysis we will first derive the form of time-periodic unidirectional solutions of the channel flow equations. In our analysis we assume that the flow rate never falls below the level at which sediment transport occurs; this assumption is usually not appropriate to gravel beds but we will use it in order to simplify the analysis. Note again however that the analysis given here could be extended to situations where transport is switched off for part of a period.

In the existing theory on bar formation little attention has been given to the question of what physical mechanism generates a particular bar wavelength. The assumption implicitly made is that in a river there are so many sources of spatial inhomogeneities that there will always be sufficient forcing at all wavelengths to excite the most unstable bar. However in less disturbed environments, e.g. in man-made channels or in laboratory experiments, the situation is less clear-cut. In fact the situation is precisely that which has been studied in some depth in the context of transition in the flow over airfoils. In that problem there is a variety of sources for the generation of instability waves. However in both wind-tunnel experiments and flight tests the particular disturbance generated depends precisely on the spatial structure of the forcing. The major point to be appreciated is that boundary layer instabilities are almost always convective in nature so an instability wave can only be generated continuously if there is a disturbance source for all time. In the context of boundary layer transition the process by which disturbances are converted into instability waves is usually called the receptivity problem; see Ruban (1984), Denier *et al.* (1991), and Duck *et al.* (1996) for a discussion of the receptivity problem for Tollmien–Schlichting and Görtler instabilities in boundary layers. In boundary layers which are convectively unstable the effect of the instability is felt only downstream of the source, in contrast to absolute instabilities (e.g. the Taylor vortex problem) where a point disturbance at some instant in time $t = 0$ spreads in all directions and persists for all positive t.

It appears that there has been no discussion of the convective/absolute instability problem for bar instabilities but it is easy to show that bar modes are in fact convective instabilities. This means that a migrating bar can only persist if there is a continuous forcing to maintain its early growth before nonlinear effects become important. Here we will show how spatial variations associated with seepage into the channel, curvature variations, local constrictions or other spatial variations can interact with flow unsteadiness to produce bars. In fact we can relate the amplitude of the bar generated to the forcing mechanism through a 'coupling coefficient' and see what kind of spatial variations are most or least efficient in the generation of bars. Thus, for example, we discuss how dykes entering the main channel or indentations made in the bank should be arranged so as to minimize or maximize bar formation.

The receptivity process is essentially a linear one and explains why only particular spatial variations are converted into migrating bars. Once a disturbance is sufficiently large nonlinear effects come into play and inhibit the exponential growth of linearly unstable modes; see CST. In the weakly nonlinear regime the multiple-scale version of the Stuart–Watson theory shows how a growing mode saturates into a finite-amplitude disturbance but in the nonlinear process the pattern associated with an unstable mode continues to propagate downstream with almost the same phase speed as that predicted by the linear theory. Thus there is no mechanism in operation which can lead to a stationary structure capable of the generation of meanders. Here we show how flow unsteadiness of small amplitude is sufficient to fix part of the disturbance field spatially. This fixing of the bar migration might then play a role in the generation of meanders.

The nonlinear theory we describe for unsteady flows might also be relevant to the role of the suppression of free alternating bars by forced bars. Tubino & Seminara (1990) have discussed the effect of a small-amplitude sinusoidal curvature distribution on the growth of free bars within the context of linear stability theory. Since the forced bar is steady on the free-bar timescale at least two nonlinear interactions involving the free and forced bars are required to produce an effect on the free bar's growth rate. This means that a small curvature distribution of size χ can at most stabilize or destabilize free bars by an amount $O(\chi^2)$. However if the basic state contains an unsteady component of size δ then the effect can be increased to $O(\chi\delta)$ so that for large unsteady variations the effect on the free bar is increased by an order of magnitude.

The procedure adopted in the rest of the paper is as follows: in § 2 we formulate the system of equations governing transport of sediment in a straight channel with the flow described by the St Venant shallow-water equations. In § 3 we discuss the linear instability of time-periodic basic states and show the effect of flow unsteadiness on the stability properties of the flow. In § 4 we investigate the receptivity problem for free bars in an unsteady flow in a channel subject to spatial variations introduced at the sidewalls. In § 5 we discuss the role of unsteadiness in the weakly nonlinear regime and give some conclusions.

2. The governing equations

We follow the formulation of the governing equations given by CST and the reader is referred to that paper for a detailed discussion of the approximations leading to the equations. We consider the flow in a rectangular channel of width $2B_0^*$ and typical depth D_0^* and scale the downstream and spanwise variables s^* and n^* on B_0^* so that

$$(s, n) = (s^*, n^*)/B_0^*.$$

If U and V are the dimensionless velocity components, scaled on a typical speed U^* in the s^*-direction, then the St Venant equations of quasi-steady shallow-water flow for a channel of sufficient width that the motion in the sidewall boundary layers can be neglected are

$$\left.\begin{aligned} U\frac{\partial U}{\partial s} + V\frac{\partial U}{\partial n} &= -\frac{\partial H}{\partial s} - \frac{\beta \tau_s}{D}, \\ U\frac{\partial V}{\partial s} + V\frac{\partial V}{\partial n} &= -\frac{\partial H}{\partial n} - \frac{\beta \tau_n}{D}, \\ \frac{\partial}{\partial s}(UD) + \frac{\partial}{\partial n}(VD) &= 0, \end{aligned}\right\} \tag{2.1}$$

where H and D are the dimensionless local water surface and depth defined by

$$(H^*, D^*) = D_0^* \left(F_0^2 H, D\right),$$

with the Froude number F_0 given by

$$F_0^2 = \frac{U^{*2}}{g D_0^*}.$$

Here g is the acceleration due to gravity. Finally the parameter β appearing above denotes the dimensionless width of the channel and is defined by

$$\beta = \frac{B_0^*}{D_0^*}, \tag{2.2}$$

whilst τ_s and τ_n are the bottom shear stresses in the s- and n-directions respectively and have been scaled by ρU^{*2}. See Johannesson & Parker (1989) for a discussion leading to the form of the St Venant equations given above.

Suppose next that the sediment flow rates in the s- and n-directions are made dimensionless using $\{((\rho_s - 1)/\rho) g d_s^*\}^{1/2}$ as an appropriate scale. Thus we write

$$(Q_s^*, Q_n^*) = d_s^* \left\{ \left(\frac{\rho_s}{\rho} - 1 \right) g d_s^* \right\}^{1/2} (Q_s, Q_n), \tag{2.3}$$

when $\rho_s$and ρ are the sediment and fluid densities, and d_s^* is a typical sediment size. If we scale time t^* by writing

$$t^* = \frac{B_0^*}{U^*} t, \tag{2.4}$$

then the continuity equation for the sediment is

$$\frac{\partial}{\partial t}\left(F_0^2 H - D\right) + Q_0 \left(\frac{\partial Q_s}{\partial s} + \frac{\partial Q_n}{\partial n} \right) = 0. \tag{2.5}$$

Here the first term represents the local time rate of change of sediment depth and Q_0 is the ratio of the scale of sediment discharge and the flow rate so that

$$Q_0 = d_s^* \left\{ \left(\frac{\rho_s}{\rho} - 1 \right) g d_s^* \right\} \Big/ [(1-p) D_0^* U^*],$$

where p is the sediment porosity.

It should be noted that the only explicit dependence on t of the equations appears in the sediment continuity equation. The system must be closed by writing down an equation which relates the sediment flow rates to the bottom stresses. Following, for example, CST we write

$$(\tau_s, \tau_n) = C(U, V)\sqrt{U^2 + V^2}, \tag{2.6}$$

and, if the bed is planar, we write

$$C = \left[\frac{1}{6 + 2.5 \ln(D/2.5 d_s)} \right]^2, \tag{2.7}$$

where $d_s = d_s^*/D_0^*$. If the bed is non-planar alternative forms of (2.7) are available but we will restrict our attention here to the planar case. The equations (2.6) and (2.7) are somewhat empirical in nature and the reader is referred to Ikeda & Parker (1985) for some motivation for the use of these equations. It remains for us to discuss an appropriate model for the transport of sediment. Fredsoe (1978) has discussed the relative importance of bedload and transport in suspension for bar instabilities. We make the approximation that transport in suspension may be neglected but note there is no intrinsic difficulty with the analysis here if we assume a more complicated model. If the sediment is transported mainly as bedload, we write

$$(Q_s, Q_n) = (\cos \varpi, \sin \varpi)\Phi, \tag{2.8}$$

where Φ is the sediment load function and ϖ is defined by

$$\sin \varpi = \frac{V}{\sqrt{U^2 + V^2}} - \frac{r}{\beta \theta^{1/2}} \frac{\partial}{\partial n} \left(F_0^2 H - D \right). \tag{2.9}$$

Here θ is the Shields parameter defined below and r is a constant usually taken to be in the range (0.3, 0.6), though CST argue that the lower value of 0.3 is more appropriate. Equation (2.9) is of course a semi-empirical relation and has been widely used to discuss the morphological evolution of cohesionless channels. If we neglect the second term on the right-hand side of (2.9) the sediment follows the flow streamlines; the second term is a model of how flow curvature alters the transport. The reader is referred to Kikkawa, Ikeda & Kitagawa (1976) and Parker & Andrews (1986) where some justification for the use of (2.9) can be found. It should also be pointed out that (2.9) is essentially a first-order linear approximation though Tubino & Seminara (1990) argue that a weakly nonlinear generalization of the equation is probably not necessary to describe small but finite-amplitude perturbations. Finally, following Chien (1954), the sediment load function Φ is given by the Meyer–Peter–Muller formula

$$\Phi = 8(\theta - 0.047)^{3/2}, \tag{2.10}$$

Here θ is the shields stress defined by

$$\theta = \frac{\tau_o^*}{(\rho_s - \rho) g d_s^*}, \tag{2.11}$$

and we see that $\theta - 0.047$ must be positive for Φ to be real. For negative values of this quantity the shear stresses at the bottom are too small to set the sediment particles into motion and there is no transport. Thus when $\theta - 0.047$ is negative we take $\Phi = 0$ but for the rest of this paper we will assume that the flow is strong enough for transport always to take place.

More precisely, we write

$$\theta = \Theta_0\sqrt{\tau_s^2 + \tau_n^2}, \tag{2.12}$$

where $\Theta_0 = \rho U^{*^2}/((\rho_s - \rho)g d_s^*)$, which, apart from a factor C, is the shields stress for a uniform flow $(U, V) = (1, 0)$ in a channel of unit depth.

2.1. *The basic states for unsteady flows*

Equations (2.1) and (2.5) support the basic steady configuration

$$(U, V, H, D) = (1, 0, H_0, 1), \tag{2.13a}$$

$$\tau_s = C_0 = \left(\frac{1}{6 + \frac{5}{2}\ln(2/5d_s)}\right)^2, \tag{2.13b}$$

$$Q_s = \Phi_0 = 8(\Theta_0 C_0 - 0.047)^{3/2}, \tag{2.13c}$$

$$H_0 = \text{constant} - \beta C_0 s. \tag{2.13d}$$

Thus, the flow is unidirectional and driven by the linear decrease along s of the water surface elevation. The sediment continuity equation is the only part of the system which has an explicit time dependence but we can in fact generate a family of unsteady solutions of the system by choosing the water surface H and velocity field to have particularly simple forms. These forms will be chosen in order to model the physical situation where the flow quantities will certainly have a time dependence on the scale during which bars develop.

We seek a solution of the equations of motion in the form of a unidirectional unsteady flow of the type

$$(U, V) = (\bar{U}(t), 0) \qquad \text{with} \qquad D = \overline{D}(t). \tag{2.14}$$

The bottom stress τ_s is then given by

$$\tau_s = \overline{C}\bar{U}^2, \ \overline{C} = \left(\frac{1}{6 + \frac{5}{2}\ln(2\overline{D}/5d_s)}\right)^2. \tag{2.15}$$

The streamwise momentum equation then yields

$$\frac{\partial \overline{H}}{\partial s} = -\frac{\beta \overline{C}\overline{U}^2}{\overline{D}(t)}. \tag{2.16}$$

We restrict attention to the case when the above ratio is independent of time which effectively enables us to find a separable solution of the equations. Thus if, for example, $\overline{U}(t)$ is specified, $\overline{D}$ is given implicitly by the equation

$$\frac{\overline{C}(t)\overline{U}^2(t)}{\overline{D}(t)} = \text{constant} = J^2. \tag{2.17}$$

The equations (2.14)–(2.17) imply that, on the bar timescale, the basic flow may be treated as a sequence of uniform flows as is the case with the kinematic wave approximations used to model flood waves. We will pay particular attention to the

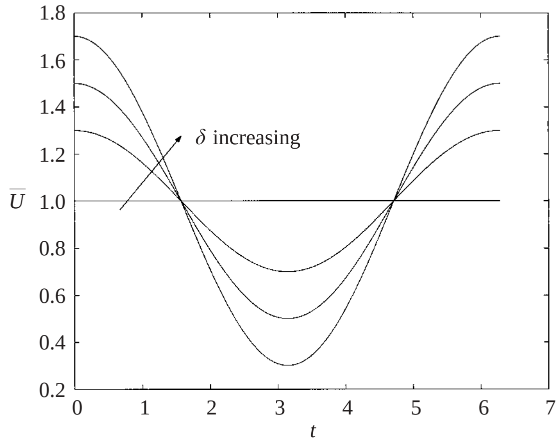

FIGURE 1. The function $\overline{U}(t)$ for $\delta = 0, 0.3, 0.5, 0.7$ with $F_0 = 0.5^{1/2}$.

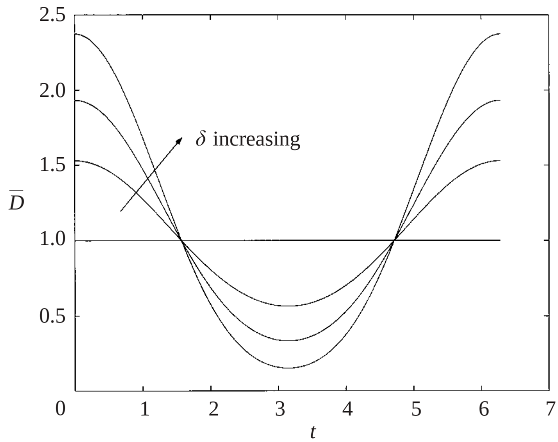

FIGURE 2. The function $\overline{D}(t)$ for $\delta = 0, 0.3, 0.5, 0.7$ with $F_0 = 0.5^{1/2}$.

case when $\overline{U}$ is a periodic function of time so, in order to normalize the flow, we suppose $\overline{U}$ has a mean value in time of unity and then take

$$J^2 = \overline{C}(\overline{D} = 1) = C_0. \tag{2.18}$$

Of course, we could alternatively specify $\overline{D}(t)$ in which case (2.17) determines the corresponding flow field explicitly. (In fact observations usually record flow rates in which case $U(t)\overline{D}(t)$ should be specified and then (2.17) can be used to find the velocity and depth.) The water surface elevation H is then given by

$$F_0^2 H = \overline{D} - F_0^2 \beta J^2 s, \tag{2.19}$$

which means (2.5) will be satisfied since $\underline{Q}_n = 0$ and $\underline{Q}_s$ will depend only on t. In figures 1, 2 and 3 we show solutions of $\overline{U}$, $\overline{D}(t)$ and $\overline{\Phi}$ when $\overline{U}$ has been chosen to be $\overline{U} = 1 + \delta \cos \omega t$, with $\delta = 0, 0.3, 0.5, 0.7$. Whilst values of δ as large as 0.7 are appropriate for gravel bed rivers, the condition that the flow never becomes slow enough for transport to cease means that our results for such large values of δ may not be physically relevant.

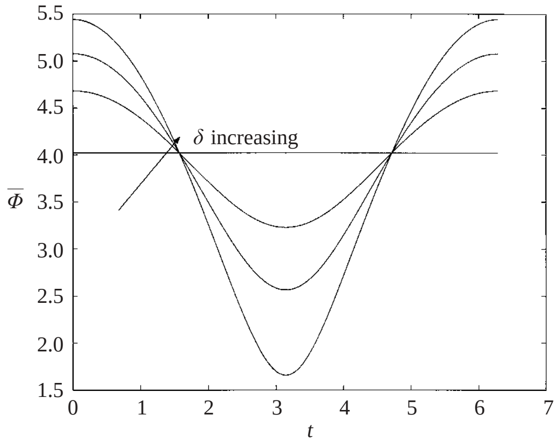

FIGURE 3. The function $\overline{\Phi}(t)$ for $\delta = 0, 0.3, 0.5, 0.7$ with $F_0 = 0.5^{1/2}$.

3. Linear instability for unsteady basic states

Previous investigations of the bar instability mechanism in rectangular channels have concentrated on steady basic states. Here, we perturb the unsteady basic states defined above by writing

$$(U, V, H, D) = (\overline{U}, 0, \overline{H}, \overline{D}) + (U_1, V_1, H_1, D_1)\mathrm{e}^{\mathrm{i}\lambda s} + \text{complex conjugate} + \ldots .$$

where U_1, V_1, H_1 and $D_1$are functions of n and t and are assumed to be small. If the corresponding perturbations in (τ_s, τ_n) and (Q_s, Q_n) are denoted by $(\tau_{s1}, \tau_{n1})\mathrm{e}^{\mathrm{i}\lambda s}$ and $(Q_{s1}, Q_{n1})\mathrm{e}^{\mathrm{i}\lambda s}$ respectively then the linearized forms of (2.1) and (2.5) are found to be

$$\left(\mathrm{i}\lambda U_1 + \frac{\partial V_1}{\partial n}\right)\overline{D} + \overline{U}\mathrm{i}\lambda D_1 = 0, \tag{3.1a}$$

$$\overline{U}\mathrm{i}\lambda U_1 + \mathrm{i}\lambda H_1 + \beta\left(\frac{\tau_{s1}}{\overline{D}} - \frac{D_1\overline{\tau_s}}{\overline{D^2}}\right) = 0, \tag{3.1b}$$

$$\overline{U}\mathrm{i}\lambda V_1 + \frac{\partial H_1}{\partial n} + \frac{\beta\tau_{n1}}{\overline{D}} = 0, \tag{3.1c}$$

$$\frac{\partial}{\partial t}\left(F_0^2 H_1 - D_1\right) + Q_0\left(\frac{\partial Q_{n1}}{\partial n} + \mathrm{i}\lambda Q_{s1}\right) = 0. \tag{3.1d}$$

Here, the stress $\overline{\tau}_s$ is the stress associated with the unperturbed basic state and therefore is given by

$$\overline{\tau}_s = \overline{C}(t)\overline{U}^2(t).$$

The perturbed stresses τ_{s1} and τ_{n1} and sediment flow rates Q_{s1}, Q_{n1} are found to be given by

$$\tau_{s1} = \overline{C}(s_1 U_1 + s_2 D_1), \tag{3.2a}$$

$$\tau_{n1} = \overline{C}V_1\overline{U}, \tag{3.2b}$$

$$Q_{s1} = \overline{\Phi}(t)\left(f_1 U_1 + f_2 D_1\right), \tag{3.2c}$$

$$Q_{n1} = \overline{\Phi}\left(V_1/\overline{U} - R\left(F_0^2 H_{1n} - D_{1n}\right)\right). \tag{3.2d}$$

Here the coefficients $s_1, s_2, f_1, f_2, R, \overline{\Phi}$ are defined by

$$s_1 = 2\overline{U},\ \ s_2 = \frac{\overline{U}^2 \partial \overline{C}/\partial D}{\overline{C}},$$
$$f_1 = \frac{2\Theta_0}{\overline{\Phi}}\overline{UC}\frac{\partial\overline{\Phi}}{\partial\theta},\ \ f_2 = \frac{\overline{U}^2\Theta_0(\partial\overline{C}/\partial D)(\partial\overline{\Phi}/\partial\theta)}{\overline{\Phi}},$$
$$R = \frac{r}{\beta\sqrt{\overline{\Theta}}},\ \overline{\Phi} = 8(\overline{\Theta} - 0.047)^{3/2},\ \ \overline{\Theta} = \theta_0\overline{CU}^2.$$

3.1. *Boundary conditions*

The banks of the river are defined by $n = \pm 1$ and at this stage they are supposed to be impermeable to the fluid and sediment so we must impose the conditions

$$V_1 = Q_{n1} = 0,\ \ n = \pm 1. \tag{3.3}$$

Following CST we notice that a solution of (3.1), (3.2) consistent with (3.3) takes the form

$$(U_1, V_1, D_1, H_1, Q_{s1}, Q_{n1}, \tau_{s1}, \tau_{n1})$$
$$= (\hat{U}_1 S_m, \hat{V}_1 C_m, \hat{D}_1 S_m, \hat{H}_1 S_m, \hat{Q}_{s1} S_m, \hat{Q}_{n1} C_m, \hat{\tau}_{s1} S_m, \tau_{n1} C_m),$$

where $C_m = \cos\frac{1}{2}mn\pi,\ \ S_m = \sin\frac{1}{2}mn\pi$ and $\hat{U}_1, \hat{V}_1$, etc. are functions only of t. After some manipulation, we find the following equation for $H_1(t)$:

$$\frac{\mathrm{d}\hat{H}_1}{\mathrm{d}t} = P_1(t, \lambda)\hat{H}_1, \tag{3.4}$$

where $P_1(t, \lambda)$ is given by

$$P_1(t) = \frac{-\Phi Q_0}{F_0^2 - \hat{D}_1}\left\{\frac{-\hat{V}_1 M}{\overline{U}} + RM^2\left[F_0^2 - \hat{D}_1\right] + \mathrm{i}\lambda[f_1\hat{U}_1 + f_2\hat{D}_1]\right\} + \frac{\mathrm{d}\hat{D}_1/\mathrm{d}t}{F_0^2 - \hat{D}_1}$$
$$= P(t) + \frac{\mathrm{d}\hat{D}_1/\mathrm{d}t}{F_0^2 - \hat{D}_1}, \tag{3.5}$$

where $M = m\pi/2$ and

$$\hat{V}_1 = \frac{-M}{\mathrm{i}\lambda\overline{U} + \beta\overline{CU}/\overline{D}},$$
$$\hat{U}_1 = \frac{\{-M(s_2\beta\overline{C} - \beta\overline{\tau}_s/\overline{D})\hat{V}_1 + \lambda^2\overline{U}\}}{\mathrm{i}\lambda\{-s_2\beta\overline{C} + \beta\overline{\tau}_s/\overline{D} + \mathrm{i}\lambda\overline{U}^2 + s_1\overline{C}\beta\overline{U}/\overline{D}\}},$$
$$\hat{D}_1 = \frac{-\overline{D}}{\mathrm{i}\lambda\overline{U}}\{\mathrm{i}\lambda\hat{U}_1 - M\hat{V}_1\}\ldots.$$

Note that a quasi-steady description of the instability leads to an instantaneous growth rate equal to $P(t)$ in (3.5) and so would be in error. It follows from (3.4) that

$$\hat{H}_1 = \frac{\hat{H}_{10}}{F_0^2 - \hat{D}_1}\exp\left\{\int^t P(t)\,\mathrm{d}t\right\},\ \ \hat{H}_{10} = \text{constant}. \tag{3.6}$$

If $\overline{U}$ and hence $\overline{H}, \overline{D}$ are periodic functions of time with period $2\pi/\omega$, then we can identify (3.6) with the Floquet solution

$$\hat{H}_1 = \mathrm{e}^{\nu t} H_p(t), \tag{3.7}$$

where H_p is periodic with period $2\pi/\omega$ and it then follows from (3.6) and (3.7) that

$$2\pi\nu = \int_0^{2\pi} \hat{P}(\tau)\,\mathrm{d}\tau \tag{3.8}$$

where $\hat{P}(\tau) = \hat{P}(\omega t) = P(t)$. Thus (3.8) shows that the Floquet exponent is independent of the driving frequency ω and that it is simply the average over a period of the instantaneous growth rate predicted by quasi-steady theory. (Note that this could have been inferred directly from (3.4) by expressing P in terms of mean and non-zero mean parts and integrating.) In the special case when the basic state is steady (3.8) reduces to the linear eigenrelation for the steady problem given by CST.

Of particular interest in the steady problem is the case when the imaginary part of the Floquet exponent, ν_i, is identically zero, because that mode has been shown by BS to lead to resonance when curvature is taken into account. In addition it is possible that a stationary bar instability in an initially straight channel might cause scouring at the banks leading to meanders. In the steady problem this possibility occurs only at very small and probably physically unrealistic values of the bar wavenumber; see BS. When δ is non-zero we see below that the possibility of a disturbance which contains a non-propagating part exists over a wide range values of the driving frequency at all values of the wavenumber. This result is perhaps the most crucial result of this section since it implies that any real flow will have oscillatory components rather than the usual steady flow assumed in the theory. In any straight channel, a fraction of any propagating bar will be fixed spatially by unsteady effects thereby opening up the possibility of meander formation by scouring. We can see this possibility by first noting that

$$\nu_i = \frac{1}{2\pi}\int_0^{2\pi} \hat{P}_i(\tau)\,\mathrm{d}\tau,$$

and point out that ν_i is independent of the frequency ω. This means that $\mathrm{e}^{\nu t} H_p$ will correspond to a disturbance which is a superposition of travelling waves and a standing wave whenever

$$\nu_i = n\omega, \ \ n = 1, 2, 3, \ldots .$$

The case when $n = 1$ corresponds to the driving frequency of the basic flow being coincident with the frequency of the bar. Note however that the frequency is a function of the unsteady flow so to ‘fix’ part of a migrating bar of the steady problem it is not quite good enough to drive the flow at the frequency of the mode of the steady problem. However our calculations discussed below show that ν_i has a weak dependence on the unsteady flow so the frequency needed for fixing a bar will be close to that of the corresponding bar of the steady problem. Thus if we calculate the neutral curve ($\nu_r = 0$) for a range of values of λ with δ fixed then at each point on the neutral curve in the (λ, β)-plane there will be an infinite number of choices for the driving frequency which will lead to a disturbance having a mean component in time at any fixed location. It is worth pointing out though that the fraction of the bar fixed by the choice of n will fall off rather quickly with n. At these points the instability will contain a part which is not propagating in the flow direction and will therefore resonate with and indeed generate curvature effects. Therefore we see that,

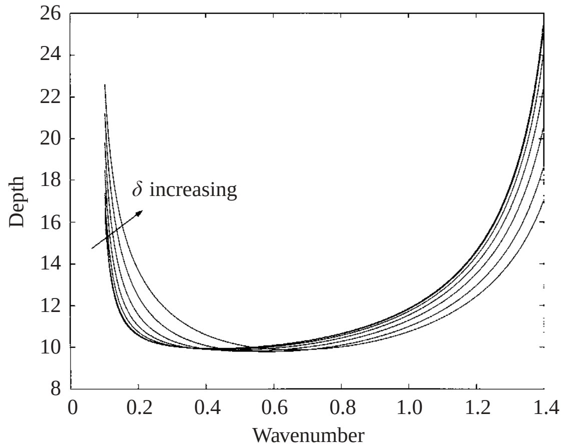

FIGURE 4. The neutral curve for $\delta = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$ with $F_0 = 0.5^{1/2}$.

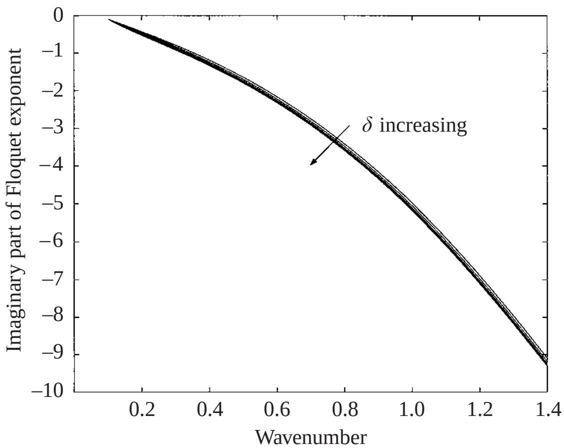

FIGURE 5. The dependence ν_i on δ for $\delta = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$ with $F_0 = 0.5^{1/2}$.

though we shall see below that the effects of the forcing on the critical width and wavelength are relatively mild, the forcing has a fundamental effect on the manner in which the instability will interact with and possibly generate weak curvature of the channel. Thus unsteady effects open up the possibility that resonant configurations will now be much more widespread than is the case when $\delta = 0$.

We concentrate on the case with $\overline{U} = 1 + \delta \cos \omega t$ and, in view of the above discussion, we can take $\omega = 1$ without any loss of generality. The neutral curves in the (β, λ)-plane can be found by fixing λ and varying β until $\nu_r = 0$. Figure 4 shows the neutral curves in the (β, λ)-plane for a range of values of δ. Figure 5 shows the dependence of ν_i on λ along the neutral curves for different values of δ. We see that the migration speed of the bar is only weakly dependent on the forcing. However, note that the time dependence of H_p interacts with the exponential time dependence of $\hat{H}_1$ to fix part of the bar at the values of ω defined above. Figure 6 shows the variation of the critical values of β and λ (scaled on their values for the steady problem) with δ. Figure 7 shows the variation in the critical value of ν_i. We observe that the flow unsteadiness has little effect on the critical value of β and ν_i but

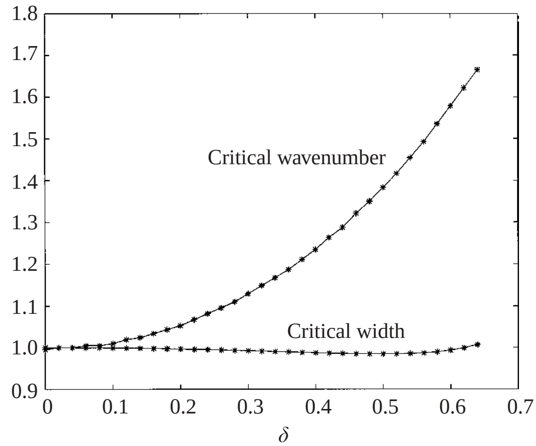

FIGURE 6. The variation of the critical widths and wavenumbers with δ for $F_0 = 0.5^{1/2}$. The results have been scaled on the critical width and wavenumber when $\delta = 0$.

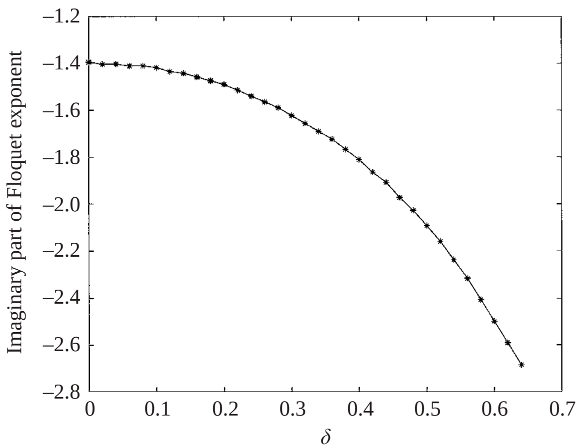

FIGURE 7. The variation of the critical value of ν_i with δ for $F_0 = 0.5^{1/2}$.

pushes the critical wavenumber to significantly higher values. Thus flow unsteadiness produces smaller-wavelength bars than are predicted by the steady theory. In the absence of controlled unsteady experiments on bar formation we cannot check the validity of this result. Indeed if we wish to check the validity of this result with reference to rivers then the situation becomes more complex. If the bars are formed at 'bankfull' conditions the most appropriate comparison of the critical wavenumber for case $\overline{U} = 1 + \delta \cos \omega t$ is with the steady flow with speed $\overline{U} = 1 + \delta$. With this in mind we repeated the calculation leading to figure 6 with $\overline{U} = (1 + \delta \cos \omega t)/(1 + \delta)$. The results are shown in figure 8 and we now observe a different trend, with the critical width significantly reduced from its value expected on the basis of a steady theory. In addition the increase in wavenumber is less than was the case without the rescaling of δ. We also note that the trends in both wavenumber and width switch at the higher values of δ.

Finally in this section we relate our linear stability theory for bars in unsteady flows to the weakly nonlinear theory of Tubino (1991). Tubino was concerned with the effect

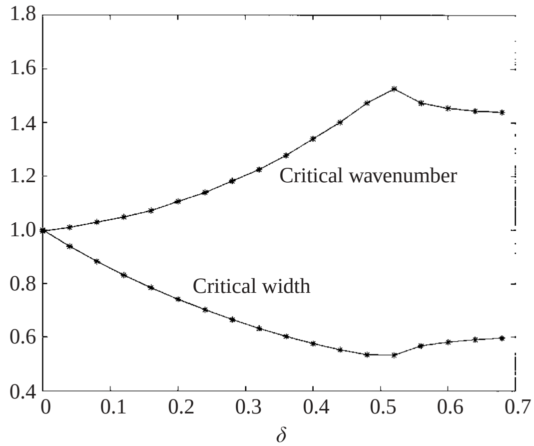

FIGURE 8. The variation of the critical widths and wavenumbers with δ for $F_0 = 0.5^{1/2}$ when $\overline{U} = (1 + \delta \cos \omega t)/(1 + \delta)$. The results have been scaled on the critical width and wavenumber when $\delta = 0$.

on the bar instability problem of a small-amplitude unsteady flow superimposed on a steady flow. Tubino was interested in the effect of flood waves so the unsteady flows considered were not periodic in time. It is straightforward to show that the flows investigated by Tubino can be found from (2.17) by expanding around a uniform steady flow. In our problem it is easy to see that an $O(\delta)$ perturbation to a steady flow will produce an $O(\delta^2)$ correction to the growth rate of a bar instability. Tubino gives a weakly nonlinear description of bar growth for small-amplitude unsteadiness comparable with the deviation of β from its critical value. The expansion procedure that he gives is valid only for unsteady flows varying on the same long timescale over which weakly nonlinear disturbances grow. Since his unsteady flow is not time periodic Tubino argues that the growth rate is altered at order δ rather than $O(\delta^2)$ as is the case here. Thus there is a fundamental difference between the periodic and non-periodic problems and we cannot make a direct comparison between the two approaches. Note however that our formulation leading to (3.4) is valid for the non-periodic problem and places no restriction on the timescale or amplitude of the unsteadiness. Thus it may well be the case that (3.4) is the appropriate formulation of the stability problem which extends the small-amplitude slow-timescale work of Tubino.

4. The receptivity problem for alternating bars

In order to motivate the perturbation analysis given below it is useful to consider the physical processes which will be described by the analysis and see how these processes generate an instability wave. Suppose in the first instance we assume a steady flow in a straight channel and now introduce a small indentation of size ε at one wall. Since the indentation is small the flow field, water depth and surface elevation will all be perturbed at $O(\varepsilon)$ and, by taking a transform in s, we can imagine that the perturbation contains all wavenumbers in the flow direction. If the width/depth ratio is comparable with its critical value then the linearized instability problem has no stationary unstable modes so the effect of the indentation will decay exponentially away from its locality. At significantly higher values of β stationary unstable modes are possible and a resonant response akin to that found by BS occurs. However, unstable stationary modes occur only at values of β typically 2–3 times the critical value and nonlinear effects will dominate the flow by that stage so that regime is probably only of academic interest. Therefore we will concentrate here on width/depth ratios near the critical value and we do not have to concern ourselves with a resonant stationary response by the flow.

Suppose then that the flow present before the indentation was introduced has a small-amplitude time-periodic component of size δ. Since the unsteadiness is small the indentation will still produce at leading order a steady perturbation to the uniform state but the nonlinear terms in the governing equations will then generate $O(\varepsilon\delta)$ terms periodic in time and containing all wavenumbers in the flow direction. At a given value of the frequency of the unsteadiness one of the wavelengths will be unstable and a growing alternate bar will be produced. Thus flow unsteadiness and spatial variations interact nonlinearly to produce instability waves. Such interactions have been investigated in great detail in the context of shear flow instabilities in both parallel and non-parallel flows. We shall now give the details of this type of interaction in the context of alternating bar instabilities.

In the situation when the basic state is steady, equation (3.4) reduces to the eigenrelation of CST by writing $\mathrm{d}/\mathrm{d}t = -\mathrm{i}\omega$ and noting that $P(t, \lambda)$ is now independent of time so that

$$-\mathrm{i}\omega = P(\lambda) \tag{4.1}$$

where we have now dropped the dependence of P on t. The reader is referred to CST and BS and the references therein for a discussion of (4.1). However, it appears that the absolute/convective nature of the instability (see, for example, Huerre & Monkewitz 1990) has not been investigated. This is a straightforward task since the eigenrelation is known explicitly for the problem under consideration and we find that (4.1) defines a convectively unstable flow. This means that a disturbance to the steady equilibrium state will be convected downstream from the position where it is imposed and that, if the flow is not continually forced, the flow will return to the equilibrium state everywhere except in a region infinitely far downstream of the initial disturbance. Thus alternating bar instabilities like, for example, Tollmien–Schlichting waves in boundary layers will only develop if they are continuously stimulated by some mechanism. Here, we shall investigate one obvious such mechanism and show how the amplitude of the transfer function associated with the induced bar may be calculated from the spectrum of the forcing function.

The first point to make is that (4.1) leads to a neutral curve in the (β, λ)-plane with $\omega_r \neq 0$ apart from one point on the left-hand branch of the curve at relatively small, physically unrealistic, values of the disturbance wavenumber. It follows that, if the unstable wavenumbers at $O(1)$ values of λ are to be stimulated, the external mechanism driving the instability must input both the time and streamwise periodic variations. We will assume that the inherent unsteadiness of the basic states discussed in the previous section is responsible for the supply of the appropriate temporal frequency. The downstream spatial variation required to interact with the flow unsteadiness can be supplied by a variety of physically relevant mechanisms.

First, we could allow the width of the channel in the unperturbed state to vary by a small amount in the s-direction. Alternatively, we could allow the channel to have a very small curvature. The simplest case is when we allow spatial variations in the

s-direction to be driven by replacing (3.3) by

$$V = \varepsilon V_F^+(s), \quad Q_n = 0, \quad n = 1,$$
$$V = 0, \qquad Q_n = 0, \quad n = -1,$$

where ε is small. These conditions can be taken to model inflow/outflow into the channel from the sidewalls. (For example a small tributary flowing into the main channel could be modelled by such a boundary condition.) We will refer to this as the seepage problem. Next we suppose that the spatial variations are driven by an indentation of one of the walls. Thus for example if we take the walls to be defined by

$$n = -1, \ n = 1 + \varepsilon F(s),$$

then, following a Taylor series expansion about $n = 1$, we find that, correct to order ε, the boundary conditions to be imposed become

$$V = \varepsilon F'(s)U, \quad Q_n = \varepsilon Q_s F'(s), \quad n = 1,$$
$$V = 0, \qquad Q_n = 0, \qquad n = -1.$$

Since the most dangerous alternating bar instability has the 'n' velocity component an even function of n, only the 'even' parts of the above conditions will lead to the generation of unstable modes, so it is sufficient for us to consider the simplified conditions:

$$V = \varepsilon V_F, \ Q_n = \varepsilon Q_F, \ n = \pm 1, \tag{4.2}$$

where $V_F = V_F^+(s)/2$, $Q_F = 0$ for the seepage problem and $V_F = (1+\delta\cos\omega t)F'(s)/2$, $Q_F = \Phi_0 F'(s)/2$ for the indentation problem. We restrict V_F and Q_F to be such that they have Fourier transforms.

We consider the case when the basic state has only a small-amplitude unsteadiness. We then write

$$\bar{U}(t) = 1 + \delta\cos\omega t, \qquad |\delta| \ll 1, \tag{4.3}$$

in which case

$$\overline{D}(t) = 1 + d\delta\cos\omega t + O(\delta^2),$$

where

$$d = \frac{2}{(1 - \overline{C'}/\overline{C})} \tag{4.4}$$

evaluated with $\overline{D}(t) = 1$. (Note that the above corresponds to equations (25), (26) of the small-amplitude unsteady flow expansion of Tubino 1991.) We then seek a solution of (2.1), (2.5) by expanding U, V, D, H in the form

$$\left.\begin{aligned} U &= 1 + \delta\cos\omega t + \varepsilon U_1(s,n) + \varepsilon\delta U_2(s,n,t) + \dots, \\ V &= \varepsilon V_1(s,n) + \varepsilon\delta V_2(s,n,t) + \dots, \\ D &= 1 + \delta d\cos\omega t + \varepsilon D_1(s,n) + \varepsilon\delta D_2(s,n,t) + \dots, \\ H &= H_0 + \varepsilon H_1(s,n) + \varepsilon\delta H_2(s,n,t) + \dots. \end{aligned}\right\} \tag{4.5}$$

Here the $O(\varepsilon)$ terms are driven by (4.2) and the $O(\delta\varepsilon)$ terms are driven by the interaction of flow unsteadiness with the spatially varying $O(\varepsilon)$ flow. It is the $O(\varepsilon\delta)$ term which will contain the unstable alternating bar modes. The shear stresses, sediment flow rates and C are then expanded as

$$\tau_s = C_0\{1 + \delta\tau_{so}\cos\omega t + \varepsilon\tau_{s1}(s,n) + \varepsilon\delta\tau_{s2}(s,n,t) + \ldots\}, \tag{4.6}$$

$$\tau_n = C_0\{\varepsilon\tau_{n1}(s,n) + \varepsilon\delta\tau_{n2}(s,n,) + \ldots\}, \tag{4.7}$$

$$\frac{C}{C_0} = 1 + \delta d\cos\omega t\hat{C}_1 + \varepsilon\hat{C}_1 D_1 + \varepsilon\delta[\hat{C}_2 D_1 D_2 + \hat{C}_1 D_2] + \ldots, \tag{4.8}$$

where

$$\hat{C}_1 = \frac{C_0'}{C_0}, \quad \hat{C}_2 = \frac{C_0''}{C_0}, \quad \frac{\tau_{s0}}{C_0} = 2\cos\omega t + d\hat{C}_1\cos\omega t,$$

$$\frac{\tau_{s2}}{C_0} = 2U_2 + 2U_1\cos\omega t + \widehat{C}_2 D_1 d\cos\omega t + C_1 D_2 + 2U_1 d\widehat{C}_1\cos\omega t + 2\cos\omega t\widehat{C}_1 D_1,$$

$$\frac{\tau_{n2}}{C_0} = V_2 + V_1\cos\omega t + V_1 d\widehat{C}_1\cos\omega t, \quad \frac{\tau_{s1}}{C_0} = 2U_1 + \widehat{C}_1 D_1, \ \frac{\tau_{n1}}{C_0} = V_1.$$

4.1. *The order-ε problem*

If we substitute the above expansions into (2.1), (2.5) and equate terms of order ε, we find that

$$\frac{\partial U_1}{\partial s} = -\frac{\partial H_1}{\partial s} - \beta C_0[\tau_{s1} - D_1], \tag{4.9a}$$

$$\frac{\partial V_1}{\partial s} = -\frac{\partial H_1}{\partial n} - \beta C_0\tau_{n1}, \tag{4.9b}$$

$$\frac{\partial U_1}{\partial s} + \frac{\partial V_1}{\partial n} + \frac{\partial D_1}{\partial s} = 0, \tag{4.9c}$$

$$\frac{\partial Q_{n1}}{\partial n} + \frac{\partial Q_{s1}}{\partial s} = 0, \tag{4.9d}$$

with

$$Q_{s1} = \Theta_0\Phi_0'\tau_{s1},$$

$$Q_{n1} = \Phi_0\left[V_1 - R\left[F_0^2 H_{1n} - D_{1n}\right]\right], \quad R = \frac{r}{\beta C_0\Theta_0}.$$

Thus the equations which determine the flow field, depth and surface elevations induced by forcing due to the indentation or seepage are homogeneous differential equations with inhomogeneous boundary conditions generated by the forcing. The homogeneous form of these equations describes the spatial evolution of small-amplitude perturbations to the steady uniform state. If we define the Fourier transform of U_1 etc. by

$$\overline{U}_1 = \int_{-\infty}^{\infty} e^{-i\lambda s} U_1\, ds$$

then, transforming the above system, and eliminating $\overline{U}_1, \overline{V}_1$ yields

$$\frac{d}{dn}\boldsymbol{Y} = B(i\lambda, 0)\boldsymbol{Y}, \quad \boldsymbol{Y} = \begin{pmatrix} \overline{D}_1 \\ \overline{D}_{1n} \\ \overline{H}_1 \\ \overline{H}_n \end{pmatrix}, \tag{4.10a, b}$$

where $\boldsymbol{B}(i\lambda, i\omega)$ is defined in Appendix A.

The unstable modes of the alternating bar problem for disturbances $\sim \mathrm{e}^{\mathrm{i}(\lambda s-\omega t)}$ satisfy the eigenrelation

$$\left|\boldsymbol{B}(\mathrm{i}\lambda, \mathrm{i}\omega) - \tfrac{1}{2}m\pi n\boldsymbol{K}\right| = 0. \tag{4.11}$$

Here $\boldsymbol{K}$ is a matrix with all elements zero except $k_{11} = k_{33} = -k_{22} = -k_{44} = 1$. We assume that β has been chosen such that $\omega = 0$ is not an eigenvalue of (4.1). The boundary conditions appropriate to (4.10) are

$$Y_2 = -\overline{V}_F\left[\frac{1}{R} + F_0^2\left(\mathrm{i}\lambda + \beta C_0^2\right)\right] + \frac{\overline{Q}_F}{R\Phi_0}, \quad Y_4 = -\left(\mathrm{i}\lambda_0 + \beta C_0^2\right)\overline{V}_F, \quad n = \pm 1. \tag{4.12}$$

It is straightforward but tedious task to solve (4.10), (4.12) and if σ_1 and σ_2 are the roots, with positive real part of

$$\begin{vmatrix} b_{21} - \sigma^2 & b_{23} \\ b_{41} & b_{43} - \sigma^2 \end{vmatrix} = 0;$$

then, after some manipulation, we find that

$$\overline{H}_1 = P_1\frac{\cosh\sigma_1 n}{\sigma_1\sinh\sigma_1} + P_2\frac{\cosh\sigma_2 n}{\sigma_2\sinh\sigma_2} = \overline{V}_F\mathscr{H}_1(n) + \overline{Q}_F\mathscr{H}_2(n),$$

$$\overline{D}_1 = \frac{P_1 b_{23}\cosh\sigma_1 n}{\left(\sigma_1^2 - b_{21}\right)\sigma_1\sinh\sigma_1} + \frac{P_2 b_{23}\cosh\sigma_2 n}{\left(\sigma_2^2 - b_{21}\right)\sigma_2\sinh\sigma_2} = \overline{V}_F\mathscr{D}_1(n) + \overline{Q}_F\mathscr{D}_2(n),$$

where P_1 and P_2 satisfy

$$P_1 + P_2 = -\left(\mathrm{i}\lambda + \beta C_0^2\right)\overline{V}_F,$$

$$\frac{b_{23}P_1}{\sigma_1^2 - b_{21}} + \frac{b_{23}P_2}{\sigma_2^2 - b_{21}} = -\left(\frac{1}{R} + F_0^2\left[\mathrm{i}\lambda + \beta C_0^2\right]\right)\overline{V}_F + \frac{\overline{Q}_F}{R\Phi_0}.$$

Finally, we note that U_1 and V_1 can then be expressed in the form

$$\overline{U}_1 = \overline{V}_F\mathscr{U}_1(n) + \overline{Q}_F\mathscr{U}_2(n), \qquad \overline{V}_1 = \overline{V}_{F1}\mathscr{V}_1(n) + \overline{Q}_{F1}\mathscr{V}_2(n),$$

together with similar expressions for $\overline{D}_1$, $\overline{H}_1$.

4.2. *The order-$\varepsilon\delta$ problem*

The nonlinear terms in the governing equations cause the order-δ unsteady flow to interact with the order-ε spatial variations produced by the indentation or seepage to produce an $\varepsilon\delta$ correction to the uniform flow. The crucial point to notice is that this correction will be time periodic and have a spatial dependence on the flow direction. At order $\varepsilon\delta$ we find the following equations for u_2, v_2 etc.:

$$\frac{\partial U_2}{\partial s} + \frac{\partial H_2}{\partial s} + \beta[\tau_{s2} - D_2]C_0 = -\cos\omega t\frac{\partial U_1}{\partial s} - \beta C_0\cos\omega t\,[2d\mathscr{D}_1 - d\tau_{s1}] + \beta D_1\tau_{s0}C_0, \tag{4.13a}$$

$$\frac{\partial V_2}{\partial s} + \frac{\partial H_2}{\partial n} + \beta C_0\tau_{n2} = -\cos\omega t\frac{\partial V_1}{\partial s} + \beta C_0\cos\omega t\, d\tau_{n1}, \tag{4.13b}$$

$$\frac{\partial U_2}{\partial s} + \frac{\partial V_2}{\partial n} + \frac{\partial D_2}{\partial s} = -\cos\omega t\frac{\partial D_1}{\partial s} - \cos\omega t d\frac{\partial U_1}{\partial s} - d\cos\omega t\frac{\partial V_1}{\partial n}, \tag{4.13c}$$

$$\frac{\partial}{\partial t}\left[F_0^2 H_2 - D_2\right] + Q_0\left[\frac{\partial Q_{n2}}{\partial n} + \frac{\partial Q_{s2}}{\partial s}\right] = 0. \tag{4.13d}$$

Q_{s2}, Q_{n2} can be found in Appendix A.

We have seen above that the $O(\varepsilon\delta)$ system is forced periodically in time with period $2\pi/\omega$ so we can seek solutions of the form

$$(U_2, V_2, H_2, D_2) = \mathrm{e}^{-\mathrm{i}\omega t}(\hat{U}_2, \hat{V}_2, \hat{H}_2, \hat{D}_2) + \text{complex conjugate}$$

where $\hat{U}_2$, $\hat{V}_2$, etc., are functions only of n and s. After taking a Fourier transform in s and some algebra we can show that the order-$\varepsilon\delta$ system may then be written in the form

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}n} = B\,(\mathrm{i}\lambda, \mathrm{i}\omega)\,\boldsymbol{X} + \begin{pmatrix} 0 \\ \varDelta_1(\lambda, n) \\ 0 \\ \varDelta_2(\lambda, n) \end{pmatrix} \overline{V}_F(\lambda) + \begin{pmatrix} 0 \\ \varDelta_3(\lambda, n) \\ 0 \\ \varDelta_4(\lambda, n) \end{pmatrix} \overline{Q}_F(\lambda), \tag{4.14}$$

with

$$\boldsymbol{X} = \begin{pmatrix} \overline{D}_2 \\ \overline{D}_{2n} \\ \overline{H}_2 \\ \overline{H}_{2n} \end{pmatrix} \tag{4.15}$$

and $\varDelta_1$, $\varDelta_2$ are given in Appendix A.

The above system must be solved subject to $X_2 = X_4 = 0, n = \pm 1$ for the seepage problem and (4.12) with $Q_F = 0$ for the indentation problem. We can therefore write $\overline{H}_2$ as

$$\overline{H}_2 = \overline{V}_F(\lambda)\mathscr{H}_3\,(\lambda, n) + \overline{Q}_F(\lambda)\mathscr{H}_4(\lambda, n). \tag{4.16}$$

The inverse transform of $\overline{H}_2$ gives

$$H_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda s} H_2\,(\lambda, n)\,\mathrm{d}\lambda. \tag{4.17}$$

The function $H_2(\lambda, n)$ will have simple poles in the complex λ-plane at any point where $\lambda = \lambda^*$ and ω satisfies (4.1) with $m = 1$. For a given value of ω and β above the critical value, there will be one such value of λ^*. For larger values of s, the right-hand side of (4.17) will be dominated by the contribution from λ^* so that

$$H_2 \sim \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\lambda^* s}(R_1(\lambda^*)\overline{V}_F(\lambda^*) + R_2(\lambda^*)\overline{Q}_F(\lambda^*)) + \text{complex conjugate} \tag{4.18}$$

where $R_1(\lambda^*)$, $R_2(\lambda^*)$ are the residues of $\mathscr{H}_3(\lambda, n)$, $\mathscr{H}_4(\lambda, n)$ at $\lambda = \lambda^*$. Equation (4.18) thus shows that the flow oscillations coupled to the spatial variations at $O(\delta)$ generate an alternating bar of amplitude $A = \varepsilon\delta(R_1(\lambda^*)\overline{V}_F(\lambda^*) + R_2(\lambda^*)\overline{Q}_F(\lambda^*))$. We note here that $R_i^*(\lambda^*)$, $i = 1, 2$, are independent of the spatial forcing in (4.18) so that the size of the bar generated is linearly proportional to a combination of the transforms of $V_{F,}$ Q_F evaluated at λ^*. Calculations show that $R_1(\lambda^*)$, $R_2(\lambda^*)$ are typically of size 10^{-1}. Equation (4.18) predicts the small-amplitude response to unsteady and spatially varying forcing of the flow. It shows how the amplitude of the response can be predicted for any small-amplitude forcing and appears to be the first quantitative prediction of bar size as a function of the flow environment. Note also that higher modes can be preferentially forced by a suitable choice of the boundary conditions but it seems reasonable to expect that in a real flow the spatial forcing will be sufficiently complex that all modes can be stimulated. In that situation the first mode will dominate because it will have the largest growth rate.

Now we shall discuss the implications of (4.18) for the two types of spatial forcing considered here; clearly the mechanism will be operational for a variety of spatial forcing mechanisms since all we need is essentially a small-amplitude spatially varying flow which can interact with an unsteady flow to generate an instability wave.

4.3. *The seepage problem*

Now let us consider the case when the spatial variations are driven by seepage so that $V_F = V_F^+(s)/2$, $Q_F = 0$. An important point to note is that for a given channel geometry the only part of the response function which depends on the nature of the seepage function V_F is the quantity $|\overline{V}_F(\lambda^*)|$. For the situation under consideration here we can now discuss how the amplitude of the bar depends on the forcing function V_F^+. Suppose for example that we are constrained to allow a given amount of fluid to enter the channel but we can vary the interval in L over which the fluid enters. Thus we take

$$V_F^+ = \frac{2}{L}, \quad -L < s < L,$$
$$V_F^+ = 0, \quad \text{otherwise}.$$

The function $|\overline{V}_F|$ is then given by

$$|\overline{V}_F| = \left|\frac{4}{\lambda^* L}\right| |\sin \lambda^* L| = 4\left|\frac{\sin J}{J}\right|$$

where $J = \lambda^* L$. Thus if L is chosen to be given by

$$L = \frac{n\pi}{\lambda^*}$$

then there is no forced alternating bar mode. This corresponds to the case when the width of the inlet channel is an integer multiple of the bar wavelength. On the other hand the largest bar is found by taking the limit $L \to 0$ so that $|\overline{V}_F| \to 4$ for each value of λ^*. We conclude that if fluid is to be discharged into a channel then the discharge should be made as wide as possible in order to minimize bar formation downstream of the inlet. Conversely if a particular migrating bar needs to be generated then the fluid should be pumped as an oscillatory flow of the desired frequency through as small an inlet as possible. This latter possibility suggests an experimental technique for bar generation not unlike the vibrating ribbon used to provoke Tollmien–Schlichting waves in boundary layers; see for example Schubauer & Skramsted (1947).

We stress that the above receptivity calculation is essentially unchanged if we assume that the order-ε forcing is produced by curvature effects, channel width variations or a variety of other means. The main point is that the nature of the interaction will always lead to an induced bar of amplitude proportional to the Fourier transform of the function defining the spatial variations of the channel evaluated at the wavenumber corresponding to the driving frequency.

4.4. *The indentation problem*

Next we consider the case when the spatial variations are driven by an indentation so that $V_F = F'(s)/2$, $Q_F = \Phi_0 F'(s)/2$. The amplitude of the induced wave is now proportional to the modulus of the transform of $F(s)$. We consider the case

$$F = \frac{2}{L}, \quad -L < s < L,$$
$$F = 0, \qquad \text{otherwise},$$

so that the area of the indentation is fixed. Following the discussion given above for the seepage problem we see that if $L = n\pi/\lambda^*$ there is no induced bar and the largest bar is induced in the limit $L \to 0$.

Finally we close this section with some comments on the relationship of this calculation to the work of Tubino & Seminara (1990) on the nonlinear interaction of free and forced bars in channel flows. In that paper the forced bar essentially corresponds to the order-ε system discussed above. However, the basic state is steady so an unstable wave cannot be directly stimulated by the forced bar. In fact two nonlinear interactions are required to force the bar mode and, in effect, Tubino & Seminar study the effect of a small forced bar on the linear instability of a free bar. The effect is order ε^2 in the absence of unsteady effects. In our calculation the effect is order ε because of the coupling of the forced mode with the flow unsteadiness.

5. Conclusion

Our concern in this paper has been the role of unsteadiness on alternating bar instabilities in sediment-carrying rivers. As a starting point it was necessary to generalize previous steady flow descriptions of the unperturbed basic state to the time-dependent regime. We found that the unsteady equations of motion for the fluid and sediment admit a separable solution with the system responding in phase at all values of the downstream variable. Mathematically this case is equivalent to taking a long-wave approximation to more general unsteady flows.

Thus the simple class of unsteady unidirectional flows derived in § 2 generalizes the unidirectional flows considered previously as the starting point for an alternating bar instability problem. Note however that the time dependence of the flow can be specified through either the depth, velocity or flux and (2.17) is then solved to determine the time dependence of the other flow quantities. In fact our approach reduces to the small-amplitude unsteady flow considered by Tubino (1991) in the linearized limit.

In our stability analysis we considered only time-periodic basic states but our analysis is relevant to quite general unsteady flows. This means that in Floquet theory we have a natural framework to define what we mean by the instability of an unsteady flow. For a more general time dependence it is necessary to look at instantaneous growth rates or possibly use an energy-based stability method to delineate stable and unstable regimes.

Figures 6 and 8 are the crucial ones that show the effect of flow oscillations on the critical width and depth. We see that the critical wavenumber increases significantly with δ and the critical depth is significantly decreased if we try to compare our results with bars generated under bankfull conditions. These predictions are difficult to compare with observations but could in principle be checked by laboratory experiments.

We also found that the introduction of flow oscillations causes the linear stability solution to have an infinite number of eigenvalues which have a steady part to their eigensolutions. The steady part of the eigenfunction does not propagate downstream and will interact with curvature variations. Indeed at these resonance points the mechanism of BS will be operational and there will therefore be a strong coupling between the alternating bar and curvature perturbations. A brief discussion of nonlinear effects is given in Appendix B: we see that in the presence of nonlinear effects, a much stronger coupling occurs and all modes will have part of their energy fixed in space by unsteadiness.

Now let us discuss the other crucial role played by flow unsteadiness in the generation of bars. We refer to the discussion of the so-called receptivity problem in §4. Here the unsteady amplitude is too small to alter the flow stability properties directly but now the unsteadiness provides a source of energy which can interact with spatial structures to produce alternating bars. It appears that this type of receptivity problem has not previously been addressed for bars even though it is now commonplace in other branches of hydrodynamic stability theory. Essentially the receptivity problem addresses questions associated with the origin of disturbances in a flow which is known to be unstable. As in many other fluid flows there exists a wealth of possible flow instabilities and the disturbance environment associated with the basic state effectively fixes the disturbance or group of disturbances to be amplified into a nonlinear state. In the flow over wings acoustic disturbances interacting with imperfections (e.g. rivets) on the wing surface internalize growing Tollmien–Schlichting waves. Experimental investigations of the receptivity problem have confirmed the theoretical work on the topic. In geomorpology this matter has not been studied.

The structure needed to generate growing disturbances in an unstable flow can be readily understood. First, if linear stability theory shows that instability first occurs through travelling wave disturbances, it is clear that the disturbance environment must introduce both spatial and temporal variations. Spatial variations can occur because of curvature variations, indentations, inflow or outflow and many other effects. However the key point is that these variations induce a steady spatially varying response which will decay exponentially away from the source if the flow is unstable only to travelling wave disturbances. Nevertheless these spatial variations are crucial because they can then interact with flow unsteadiness to produce growing travelling waves. Thus in the neighbourhood of say an indentation the leading-order response of the flow is to produce a localized spatial structure near the indentation. At higher order this structure interacts with flow unsteadiness to produce travelling wave disturbances which grow exponentially as they propagate away from the indentation. Thus, far downstream the flow contains no record of the spatial variation of the flow which selectively amplified the disturbance in question.

The major results from the receptivity work concern the relationship of the amplitude of the induced instability wave to the spatial variations which induced it. Our work shows clearly how the size of an induced bar can be calculated from the signature of the spatial structure inducing it. Thus we have given an explicit formula for the amplitude of the bar generated by different types of spatial inhomogeneities. It remains to be seen whether the mechanism we have described can be verified by careful laboratory experiments. If the receptivity process we have described is indeed operational in rivers then it will provide the means to understand how bar wavelengths and amplitudes are selected at any point along the river. In fact it is straightforward to formulate a strategy to determine the amplitudes of a finite band of unstable wavenumbers. Suppose that we know all the flow properties of a given river and at the given value of β there is a range of unstable wavenumbers. If we choose K wavenumbers spanning this interval and choose $K+1$ indentations then we can choose the indentation amplitudes $\{s_k\}$ to minimize or maximize the amplitudes of the different modes stimulated by the indentations; see for example Balakumar & Hall (1999) for related methods applied to boundary layer control by suction. This would provide a method to estimate the best and worst case scenarios for the generation of bars by random indentations.

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Appendix A

The matrix $\boldsymbol{B}(\mathrm{i}\lambda, \mathrm{i}\omega)$ appearing in (4.10) is defined by $b_{ij} = 0$ apart from

$$b_{11} = 1, \quad b_{21} = a_7, \quad b_{23} = a_8, \quad b_{34} = 1, \quad b_{41} = a_4, \quad b_{43} = a_5,$$

with

$$a_1 = 1 + \frac{2\beta C_0^2}{\mathrm{i}\lambda}, \quad a_2 = \beta C_0(C_0\widehat{C}_1 - 1) - \left(\mathrm{i}\lambda + 2\beta C_0^2\right),$$
$$a_3 = \mathrm{i}\lambda + \beta C_0^2, \quad a_4 = -\frac{a_2 a_3}{a_1}, \quad a_5 = -\frac{\mathrm{i}\lambda a_3}{a_1},$$
$$a_6 = \Phi_0\left(RF_0^2 + \frac{1}{a_3}\right),$$
$$a_7 = -\frac{1}{R\Phi_0}\left\{\frac{\mathrm{i}\omega}{Q_0} - a_4 a_6 + C_0\Theta_0\Phi_0'\left[\widehat{C}_1\mathrm{i}\lambda - \frac{2\beta C_0}{a_1}(C_0\widehat{C}_1 - 1)\right]\right\},$$
$$a_8 = -\frac{1}{R\Phi_0}\left\{-\frac{\mathrm{i}\omega}{Q_0} - a_5 a_6 - \frac{2}{a_1}\mathrm{i}\lambda\Theta_0\Phi_0' C_0\right\}.$$

The functions Q_{s2}, Q_{n2} appearing in (4.13) are given by

$$Q_{s2} = C_0^2\Theta_0^2\Phi_0''\tau_{s0}\tau_{s1}\cos\omega t + C_0\Theta_0\Phi_0'\tau_{s2},$$

$$Q_{n2} = \Phi_0\left\{V_2 - V_1\cos\omega t - R\left[F_0^2 H_{2n} - D_{2n}\right] + \frac{R}{2}\tau_{s0}\left(F_0^2 H_{1n} - D_{1n}\right)\right.$$
$$\left. + \frac{\Phi_0'}{\Phi_0}\tau_{s0}C_0\Theta_0\left[V_1 - R\left(F_0^2 H_{1n} - D_{1n}\right)\right]\right\},$$

The quantities Δ_1, Δ_2 in (4.15) are defined by

$$\Delta_1 = -\frac{1}{R\Phi_0}\left(\frac{\Phi_0 F_2'}{a_3} - a_6 F_1 + 2C_0\Theta_0\Phi_0'\frac{F_1}{a_1} - F_4\right), \quad \Delta_2 = \frac{a_3}{a_1}(F_1 - a_1 F_3) + F_2',$$
$$F_1 = \mathscr{U}_1\left\{-2\beta C_0^2 - \mathrm{i}\lambda + 2\beta C_0^2 d - 2\beta C_0^2 d\widehat{C}_1\right\}$$
$$+\mathscr{D}_1\beta C_0\{2 - d\widehat{C}_2 - 2\widehat{C}_1 + dC_0\widehat{C}_1 + d\widehat{C}_1 - 2d\}.$$
$$F_2 = -\mathscr{V}_1\left\{\mathrm{i}\lambda + \beta C_0^2[1 + d\widehat{C}_1 - d]\right\},$$
$$F_3 = -\left\{\mathrm{i}\lambda\mathscr{D}_1 + \mathrm{i}\lambda d\mathscr{U}_1 + d\frac{\partial\mathscr{V}_1}{\partial n}\right\},$$

$$F_4 = -\Phi_0 Q_0\{-\mathscr{V}_{1n}\}\left[1 - \frac{\Phi_0'}{\Phi_0}C_0\Theta_0(2+d\widehat{C}_1)\right]$$
$$- RC_0\left(F_0^2\mathscr{H}_{1nn} - \mathscr{D}_{1nn}\right)\left(\frac{\Phi_0'}{\Phi_0}\Theta_0 + \frac{1}{2}\right)(2+d\widehat{C}_1)$$
$$- C_0Q_0\Theta_0\Phi_0\{\mathrm{i}\lambda\mathscr{D}_1(\widehat{C}_2 + 6\widehat{C}_1) + \mathrm{i}\lambda\mathscr{U}_1(10+2d\widehat{C}_1)\}$$

and Δ_3, Δ_4 are found from the expressions for Δ_1, Δ_2 respectively by replacing $\mathscr{H}_1$, $\mathscr{V}_1$, $\mathscr{D}_1$, $\mathscr{U}_1$ by $\mathscr{H}_2$, $\mathscr{V}_2$, $\mathscr{D}_2$, $\mathscr{U}_2$ respectively.

Appendix B

It was shown by BS that the point on the neutral curve where ω_n vanishes (say $\lambda = \lambda_{n0}$) has a particular significance when the river is allowed to have a small curvature. This problem is described by the equations here, modified to allow for the effect of secondary flows induced by curvatures on the sedimentation process. If χ is a measure of the curvature, then the flow driven by the curvature can be found by expanding U in the form

$$U = 1 + \chi^{1/3}U_1(n,s) + \cdots .$$

With $\beta = \beta_n + \chi^{2/3}\widehat{\beta}$ and it is found that U_1 and the corresponding flow property perturbations at order χ satisfy the steady linearized version of the linear system found in § 3 with $\delta = 0$ but forced by inhomogeneous curvature-driven terms. The resonance occurs when the curvature has in its Fourier decomposition the wavenumber λ_{n0} and following Hall & Walton (1978) it can be shown that the imperfect bifurcation problem required to smooth out this singular behaviour leads to an amplitude equation of the form

$$\frac{\mathrm{d}A}{\mathrm{d}T} = \widehat{\beta}h_1A + h_2A\,|A|^2 + h_3.$$

Thus the curvature effect leads to the term h_3 above and this unfolds the bifurcations into a smooth transition to a steady finite-amplitude wave selected by the curvature.

Next we show that when $\delta = O(\chi^{1/2})$, the above equation may be generalized to describe curvature, bar and flow unsteadiness interacting to select a particular bar structure driven by curvature and unsteadiness, with the curvature reinforced by the bar and the flow unsteadiness. Now suppose that δ, the amplitude of the flow unsteadiness, is small and that the unsteadiness is periodic in time with period $2\pi/\omega_n$. The parameters δ and χ are taken to be such that $\delta = \chi^{1/2}$ so we formally write $\delta = D\chi^{1/2}$ and consider the limit $\chi \to 0$, with D held fixed. We now expand U in the form

$$U = 1 + \left[D\mathrm{e}^{\mathrm{i}\omega t}\chi^{1/2} + \text{complex conjugate}\right]$$
$$+ \left[A(T)\chi^{1/2}\mathrm{e}^{\mathrm{i}(\lambda_n s - \omega_n t)}U_1(\eta) + \text{complex conjugate}\right]$$
$$+ \left[\chi U_2(\eta)\mathrm{e}^{\mathrm{i}\lambda_n s}J + \text{complex conjugate}\right] + \cdots.$$

The first term above corresponds to the unperturbed steady flow whilst the second term is the small-amplitude oscillation imposed on the later flow. The third term is a finite-amplitude disturbance appropriate to channel widths β such that

$$\beta = \beta_n + \chi\widehat{\beta}_n$$

with $T = \chi t$. The interaction associated with the second and third terms in the expansions of U produces an order-χ term which reinforces the curvature-driven term of order χ. However, the interaction of the second and fourth terms above produces a resonant response because it generates a neutrally stable wave $\sim e^{i(\lambda_n s - \omega_n t)}$. Following the procedure of Hall & Walton (1978) we find that A will satisfy an equation of the form

$$\frac{dA}{dT} = \widehat{\beta}_n h_1 A + h_2 A |A|^2 + h_4 \overline{D} J$$

where h_4 is a constant. The constant J is determined by the solution of the order-χ problem, which leads to an equation of the form

$$J = J_0 + h_5 A \overline{D}$$

where J_0 is a constant determined by the scaled channel curvature, and h_5 is another constant. The above equations can be combined to give

$$\frac{dA}{dt} = A\{\widehat{\beta}_n h_1 + h_4 h_5 |D|^2\} + h_2 A |A|^2 + h_4 J_0 \overline{D}.$$

Thus curvature and flow unsteadiness combine to give an imperfect bifurcation problem for the alternating bar. The bar then drives J through the equation above so that the bar modifies the curvature. For large values of $\widehat{\beta}_n^{1/2}$ we have a large response to the stationary part of the disturbed flow. Thus, flow unsteadiness couples with migrating bars to produce a relatively large-amplitude stationary s-periodic structure which might play a role in bend formation.

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